Polar Representations of Quaternions

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Complex numbers have a polar representation. Three complex numbers that share the same real number are subgroups of a quaternion. Therefore a polar representation of a quaternion must exist. The amplitude is the absolute value of the whole quaternion. The imaginary number i expands to i, j, k for quaternions. The remaining question is how to handle the angle. Two ways work. The first is to take the inverse cosine of the scalar over the absolute value of the quaternion. The second method takes the inverse tangent of the absolute values of the 3-vector over the scalar. A right triangle is animated so the connection between velocity and the polar representation is more apparent.

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Calculate the various parts that go in to the polar representation:

calc

```
A = q[1, 2, 3, 4];
```

calc

```
absA = abs[A]
```

calc

$\sqrt{30}$

calc

```
theta = ArcCos[scalar[A] / abs[A]]
```

calc

$\text{ArcCos}\left[\frac{1}{\sqrt{30}}\right]$

calc

```
ivec = qvector[A / norm[qvector[A]]]
```

calc

$\left\{\left\{0, -\frac{2}{29}, -\frac{3}{29}, -\frac{4}{29}\right\}, \left\{\frac{2}{29}, 0, -\frac{4}{29}, \frac{3}{29}\right\}, \left\{\frac{3}{29}, \frac{4}{29}, 0, -\frac{2}{29}\right\}, \left\{\frac{4}{29}, -\frac{3}{29}, \frac{2}{29}, 0\right\}\right\}$

calc

```
ivecabs = qvector[A / abs[qvector[A]]]
```

calc

$\left\{\left\{0, -\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, -\frac{4}{\sqrt{29}}\right\}, \left\{\frac{2}{\sqrt{29}}, 0, -\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}\right\}, \left\{\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}, 0, -\frac{2}{\sqrt{29}}\right\}, \left\{\frac{4}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}, 0\right\}\right\}$

$$A = (1, 2, 3, 4) \qquad (1)$$

$$\left| A \right| = \sqrt{30} \qquad (2)$$

$$\theta = \arccos\left(\frac{A + A^*}{2 \mid A \mid}\right) = \arccos\left(\frac{\text{scalar }(A)}{\mid A \mid}\right) = \arccos\left(\frac{1}{\sqrt{30}}\right) \qquad (3)$$

$$\vec{I} = \frac{\text{vector }(A)}{\mid \text{vector }(A) \mid} = \left(0, \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right) \qquad (4)$$

The polar representation of a quaternion is the magnitude of the quaternion times the exponential of angle theta in the I direction:

calc

```
absA qexp[theta ivecabs].{1, 0, 0, 0}
```

calc

```
{1, 2, 3, 4}
```

polar rep makes same value

$$\left| A \right| \exp\left(\theta \vec{I}\right) = (1, 2, 3, 4) \tag{5}$$

Bingo, bingo, exactly right. Write it out:

polar representation

$$q = \sqrt{q\, q^*} \exp\left(\arccos\left(\frac{q + q^*}{2\sqrt{q\, q^*}}\right) \frac{q - q^*}{\sqrt{-(q - q^*)^2}}\right) \tag{6}$$

A short paper by Drahoslava Janovska ́, and Gerhard Opfer gave used an Arctan instead of an Arccos, and a different angle for the polar representation (available at http://www.waset.org/journals/waset/v47/v47-29.pdf). Try their suggestion at the bottom of the polar decomposition.

calc

```
α = ArcTan[abs[qvector[A]] / scalar[A]]
```

calc

$$\text{ArcTan}\left[\sqrt{29}\right]$$

polar values using the vector

$$\alpha = \arctan\left(\frac{\left| A - A^* \right|}{\left| A + A^* \right|}\right) = \arctan\left(\sqrt{29}\right) \tag{7}$$

calc

```
(abs[A] qexp[α ivecabs]).{1, 0, 0, 0}
```

calc

```
{1, 2, 3, 4}
```

polar alternative rep has same value

$$\left| A \right| \exp\left(\alpha \vec{I}\right) = (1, 2, 3, 4) \tag{8}$$

Bingo, bingo.

Alternative polar representation

$$q = \sqrt{q\, q^*} \exp\left(\arctan\left(\frac{\sqrt{-(q - q^*)^2}}{(q + q^*)}\right) \frac{q - q^*}{\sqrt{-(q - q^*)^2}}\right) \tag{9}$$

What is the difference between the arctan and the arccosine? A cosine is the near side over the hypotenuse. A tangent is the far over the near side. The choice for the definiton of θ reflects the difference between the two trig functions.

$$N\left[\text{ArcCos}\left[\frac{1}{\sqrt{30}}\right]\right]$$

$$N\left[\text{ArcTan}\left[\sqrt{29}\right]\right]$$

```
1.38719

1.38719
```

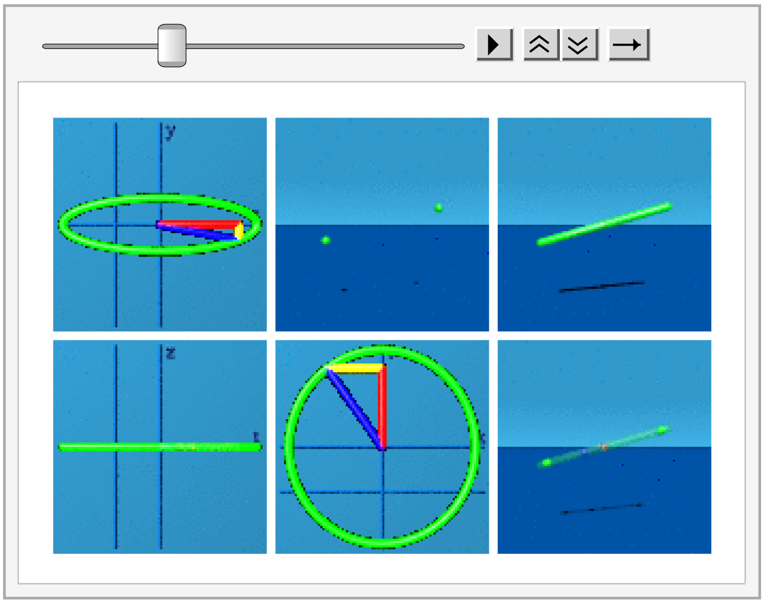
Why does the polar representation matter? The euclidean representation are 4 independent numbers: think of an unending grid. The polar representation connects the numbers together, but what does it mean? Consider the polar representation of events in spacetime, with time as the scalar and displacement as the 3-vector. The |A| part is positive definite, the distance covered in spacetime. The exponential is an angle times an imaginary number. For a quaternion, the imaginary number is a 3-vector, so can point in an arbitrary direction in 3D space. The ratio of change in space to change in time is the velocity.

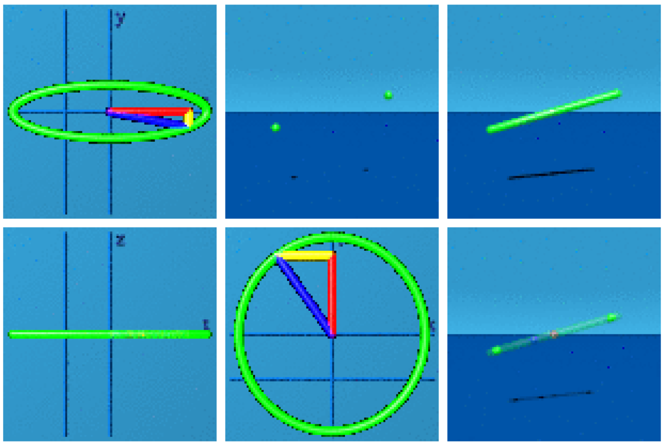
All of the trig relationships are about right triangles inside a unit circle. Both the unit circle and a right triangle can be animated:

calc

```
TimelikeRightTriangle = Import[
   "http://dl.dropbox.com/u/2779019/qqft/timelike_future.povray.100.1005.animation.scan.
     gif"];
ListAnimate[TimelikeRightTriangle]
```

calc





Circles and right triangles look like straight lines in spacetime, although they can "look right" in the three complex planes. The interval between the origin and the last blue event is timelike. The slope of the blue line is fairly close to 45 degrees so this represents a fast moving particle. A classical particle would make a very narrow triangle. For this future spacelike right triangle, the observer is in red, never moving, while the particle in motion gets farther away. When the particle in motion reaches the unit circle, there is what I call a "bling" when many events appear on the screen for a moment which connects the observer to the particle in motion. Two of the three sides of the right triangle could be transversed by real particles.

A triangle formed from spacelike events appears similar, only the interpretation changes. Only one of the world lines could be done by a real particle. A triangle in 3D space appears for only an instant, and has nothing to do with the polar representation because there is no change in time.

post, calc

■ Initialization functions

calc

Define a function for quaternions using its matrix representation.

calc

$$q[t\_, x\_, y\_, z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

calc

```
qscalar[q1_] :=
    q[q1[[1, 1]], 0, 0, 0]
```

calc

```
scalar[q1_] :=
    q1[[1, 1]]
```

calc

The vector part of a quaternion, qVector[q] = (0, V)

calc

```
qvector[q1_] :=
    q[0, q1[[2, 1]], q1[[3, 1]], q1[[4, 1]]]
```

calc

The conjugate of a quaternion it its transpose, qconj[q] = (t, -V)

calc

```
qconj[q1_] :=
    Transpose[q1]
```

calc

The square of the norm of a quaternion is the conjugate of the quaternion times the quaternion, norm[q]= t^2 + V.V

calc

```
norm[q1_] :=
        (Transpose[q1] . q1)[[1, 1]]
```

calc

The norm of the vector, normVector[q] = V.V

calc

```
normVector[q1_] :=
        q1[[2, 1]]^2 + q1[[3, 1]]^2 + q1[[4, 1]]^2
```

calc

```
sqrtNormVector[q1_] :=
        Sqrt[q1[[2, 1]]^2 + q1[[3, 1]]^2 + q1[[4, 1]]^2]
```

calc

The absolute value.

calc

```
abs[q1_] := Sqrt[(Transpose[q1].q1)[[1, 1]]]
```

calc

The exponential, qexp = (Exp[t] Cos[|V|], Exp[t] Sin[|V|])

calc

```
qexp[q1_] :=
Module[{vnorm},
        vnorm = sqrtNormVector[q1];
        If[vnorm != 0, vfactor = Exp[q1[[1, 1]]] Sin[vnorm] / vnorm,
            vfactor = 0];
        q[Exp[q1[[1, 1]]] Cos[vnorm], vfactor q1[[2, 1]],
                vfactor q1[[3, 1]], vfactor q1[[4, 1]]] ]
```