Lorentz Boosts with Quaternions

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A method for using hyperbolic sines and cosines in a real-valued quaternion to generate a Lorentz boost along an axis is shown. Do precisely what one does for a 3D spacial rotation as a first step, substituting the hyperbolic for regular trig functions, B' = H B H*. That creates four terms that are need, adding two extra terms and containing two omissions. A difference between the same three matrices does the job: B' = H B H* + ((H H B)* - (H* H* B)*)/2. The inverse transform is created by changing the conjugates on the hyperbolic quaternions. Boosts represented with quaternions must form a group, but it is not compact because the operator uses both addition and multiplication.

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Start by understanding rotations around the x axis with quaternions.

calc

```
cs = q[C, S, 0, 0];
u = q[Cos[α], Sin[α], 0, 0];
R = q[ct, x, y, z];
Simplify[cs.R.qconj[cs]].{1, 0, 0, 0}
Simplify[u.R.qconj[u]].{1, 0, 0, 0}
```

calc

$$\left\{ct\left(C^2+S^2\right),\ \left(C^2+S^2\right)x,\ C^2 y - S^2 y - 2\,C\,S\,z,\ 2\,C\,S\,y + C^2 z - S^2 z\right\}$$

calc

$$\{ct,\ x,\ y\,\mathrm{Cos}[2\,\alpha] - z\,\mathrm{Sin}[2\,\alpha],\ z\,\mathrm{Cos}[2\,\alpha] + y\,\mathrm{Sin}[2\,\alpha]\}$$

rotations using sines cosines and quaternions

$$(c\,t', x', y', z') = (\cos(\alpha),\ \sin(\alpha), 0, 0)\,(c\,t, x, y, z)\,(\cos(\alpha),\ -\sin(\alpha), 0, 0) \tag{1}$$

$$= \left(\left(\sin^2(\alpha)+\cos^2(\alpha)\right)c\,t, \left(\sin^2(\alpha)+\cos^2(\alpha)\right)x,\right.$$

$$\left.\left(\cos^2(\alpha)-\sin^2(\alpha)\right)y - 2\sin(\alpha)\cos(\alpha)\,z, \left(\cos^2(\alpha)-\sin^2(\alpha)\right)z + 2\sin(\alpha)\cos(\alpha)\,y\right)$$

$$= \ (c\ t, x, \cos(2\ \alpha)\ y - \sin(2\alpha)\ z\ , \cos(2\alpha)\ z + \sin(2\alpha)\ y)$$

The first two terms, $c\,t$ and x, are unaltered due to a vector identity. The other two terms use the double angle identities to mix the y and z as happens for a rotation around the x axis. Why the factor of 2? My explanation is that one is using the angle and its mirror reflection via the conjugate operator. This pattern of what gets changed versus what does not is flipped for a boost along the x axis. In that case, it is the y and z that are not altered, and the t and x that mix it up with hyperbolic sines and cosines. Start out by using the same structure used for rotations, but substituting hyperbolic trig functions:

calc

```
chsh = q[Ch, Sh, 0, 0];
h = q[Cosh[α], Sinh[α], 0, 0];
Simplify[chsh.R.qconj[chsh]].{1, 0, 0, 0}
Simplify[h.R.qconj[h]].{1, 0, 0, 0}
Simplify[h.R.qconj[h]].{1, 0, 0, 0} /.
  {Cosh[2 α] -> γ, Cosh[α] Sinh[α] -> γβ, Sinh[2 α] -> γβ}
```

calc

$$\left\{ct\left(Ch^2+Sh^2\right),\ \left(Ch^2+Sh^2\right)x,\ Ch^2 y - Sh^2 y - 2\,Ch\,Sh\,z,\ 2\,Ch\,Sh\,y + Ch^2 z - Sh^2 z\right\}$$

calc

$$\{ct\,\mathrm{Cosh}[2\,\alpha],\ x\,\mathrm{Cosh}[2\,\alpha],\ y - 2\,z\,\mathrm{Cosh}[\alpha]\,\mathrm{Sinh}[\alpha],\ z + y\,\mathrm{Sinh}[2\,\alpha]\}$$

calc

$$\{ct\,\gamma,\ x\,\gamma,\ y - 2\,z\,\gamma\beta,\ z + y\,\gamma\beta\}$$

partial boosts using hyperbolic sines cosines and quaternions

$$(c\,t', x', y', z') = (\cosh(\alpha),\ \sinh(\alpha), 0, 0)\,(c\,t, x, y, z)\,(\cosh(\alpha),\ \sinh(\alpha), 0, 0)^* \tag{2}$$

$$= \left(\left(\sinh^2(\alpha)+\cosh^2(\alpha)\right)c\,t, \left(\sinh^2(\alpha)+\cosh^2(\alpha)\right)x,\right.$$

$$\left.\left(\cosh^2(\alpha)-\sinh^2(\alpha)\right)y - 2\sinh(\alpha)\cosh(\alpha)\,z, \left(\cosh^2(\alpha)-\sinh^2(\alpha)\right)z + 2\sinh(\alpha)\cosh(\alpha)\,y\right)$$

$$= \ (\cosh(2\alpha)\ c\ t, \cosh(2\alpha)\ x, y - \sinh(2\alpha)\ z, z + \sinh(2\alpha)\ y)$$

$= (\gamma\, c\, t, \gamma x, y - \gamma\beta z, z + \gamma\beta\, y)$

Four of these terms are what is needed for a boost along the *x* axis: the γct, γx, *y* and *z*. The gamma beta terms are missing where they are needed (*t, x*), and appear where they should not (*y, z*). To fix these issues, a quaternion must have four gamma betas with just the right combinations of signs. The product must mix the *c t* with the *x*, and the *y* with the *z*. That will not happen with the position quaternion in the middle, so put it on an end:

calc

```
Simplify[h.h.R].{1, 0, 0, 0}
```

calc

```
{ct - 2 x Cosh[α] Sinh[α], x + ct Sinh[2 α], y - 2 z Cosh[α] Sinh[α], z + y Sinh[2 α]}
```

$$(\cosh(\alpha),\ \sinh(\alpha),\ 0,\ 0)^2\,(t,\ x,\ y,\ z) \tag{3}$$

$= (1, \sinh(2\alpha), 0, 0)(c\, t, x, y, z)$

$= (c\, t - \sinh(2\alpha)\, x, x + \sinh(2\alpha)\, c\, t, y - \sinh(2\alpha)\, z, z + \sinh(2\alpha)\, y)$

$= (c\, t - \gamma\beta\, x, x + \gamma\beta\, c\, t, y - \gamma\beta\, z, z + \gamma\beta\, y)$

This product has several problems. The hyperbolic sign in the first term has the right sign, but not in the second. This can be corrected by taking the conjugate. The *t, x, y*, and *z* terms generated by the unitary scalar need to be eliminated. The triple product needs to be twisted just right to keep what is needed. The hyperbolic rotation used one term conjugated, the other not, so subtract the conjugate squared:

calc

```
Simplify[1/2 (qconj[q[Cos[a], Sin[a], 0, 0].q[Cos[a], Sin[a], 0, 0].R] -
    qconj[q[Cos[a], -Sin[a], 0, 0].q[Cos[a], -Sin[a], 0, 0].R])].{1, 0, 0, 0}
```

calc

```
{-2 x Cos[a] Sin[a], -2 ct Cos[a] Sin[a], z Sin[2 a], -2 y Cos[a] Sin[a]}
```

calc

```
Simplify[1/2 (qconj[h.h.R] - qconj[qconj[h].qconj[h].R])].{1, 0, 0, 0}
```

calc

```
{-x Sinh[2 α], -ct Sinh[2 α], z Sinh[2 α], -y Sinh[2 α]}
```

$$\frac{1}{2}\left(\left((\cosh(\alpha),\ \sinh(\alpha),\ 0,\ 0)^2\,(c\,t,\ x,\ y,\ z)\right)^* - \left((\cosh(\alpha),\ \sinh(\alpha),\ 0,\ 0)^{*2}\,(c\,t,\ x,\ y,\ z)\right)^*\right) \tag{4}$$

$= \frac{1}{2}\left(\left((1,\ \sinh(2\,\alpha), 0, 0)^2\,(c\,t,\ x,\ y,\ z)\right)^* - \left((1,\ -\sinh(2\,\alpha), 0, 0)^2\,(c\,t,\ x,\ y,\ z)\right)^*\right)$

$= (-\sinh(2\alpha)\, x, -\sinh(2\alpha)\, c\, t, \sinh(2\alpha)\, z, -\sinh(2\alpha)\, y)$

$= (-\gamma\beta\, x, -\gamma\beta\, y, \gamma\beta\, z, -\gamma\beta\, y)$

This is exactly what is needed! Combine:

calc

```
Simplify[QBx = h.R.qconj[h] + 1/2 (qconj[h.h.R] - qconj[qconj[h].qconj[h].R])].{1, 0, 0, 0}
```

calc

```
{ct Cosh[2 α] - 2 x Cosh[α] Sinh[α], x Cosh[2 α] - 2 ct Cosh[α] Sinh[α], y, z}
```

$$b' = h\, b\, h^* + \frac{1}{2}\left((h\, h\, b)^* - (h^*\, h^*\, b)^*\right) \tag{5}$$

$= (\cosh(2\alpha)\, c\, t - \sinh(2\alpha)\, x, \cosh(2\alpha)\, x - \sinh(2\alpha)\, c\, t, y, z)$

$= (\gamma c\, t - \gamma\,\beta\, x, \gamma x - \gamma\,\beta\, c\, t, y, z)$

Show this does indeed preserve the interval.